

# B-SPLINE CURVES AND SURFACES AS A MINIMIZATION OF QUADRATIC OPERATORS

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ABSTRACT. The goal of this short note is to prove that every b-spline curve or surface (generated by uniform knots, without multiplicity) may be defined as minimum of positive quadratic operator.

## 1. INTRODUCTION

The b-spline curves and surfaces are an essential tool in many engineering software for design and visualization – for example ANSYS, RFEM 3D, etc. So, it is necessary to have in these applications many different methods to construct b-spline elements. Also see [11], [12], and references therein.

We will prove that any b-spline curve or surface minimizes positive quadratic operator: appropriate moving least-square error.

Let us mark that different approaches in moving least-squares method are used by Shepard – computer software SYMAP (Harvard Laboratory for Computer Graphics), Lancaster in 1979, and the works of D. Levin in 1999, see [8]. In [10] it has been shown that moving least-squares method is an adequate mathematical tool for determining diesel-fuel cetane-number (or cetane-index) from easily available physical properties of fuels.

In this section we will remind the definition of b-splines generated by control points in  $\mathbb{R}^{d+1}$  and definition of moving least-squares approximation for a given data set  $\{(\mathbf{x}_i, f(\mathbf{x}_i)) : \mathbf{x}_i \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$ .

## 2. PRELIMINARIES

**2.1. b-Splines.** Let  $\{\mathbf{p}_i \in \mathbb{R}^{d+1} : i = 0, \dots, n\}$  be a set of  $n + 1$  (control) points.

Let  $r$  be an integer,  $1 \leq r \leq n + 1$  (the order of spline).

We will use uniform knots, without multiplicity:  $t_i = i, i = 0, \dots, n + r$ .

Using Cox-de Boor recursion formula (see [3], [4]), let us define the following basis functions:

$$B_{i,1}(t) = \begin{cases} 1, & \text{if } t_i \leq t < t_{i+1}, \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

for  $0 \leq i \leq n + r - 1$ ; and

$$\begin{aligned} B_{i,j}(t) &= \frac{t - t_i}{t_{i+j-1} - t_i} B_{i,j-1}(t) + \frac{t_{i+j} - t}{t_{i+j} - t_{i+1}} B_{i+1,j-1}(t) \\ &= \frac{t - i}{j - 1} B_{i,j-1}(t) + \frac{i + j - t}{j - 1} B_{i+1,j-1}(t), \end{aligned} \quad (2)$$

for  $2 \leq j \leq r$ ,  $0 \leq i \leq n + r - j$ .

The b-spline curve of order  $r$  is defined as a linear combination of control points  $\mathbf{p}_i$ :

$$\gamma(t) = \sum_{i=0}^n B_{i,r}(t) \mathbf{p}_i, \quad t \in [t_{r-1}, t_{n+1}]. \quad (3)$$

## 2.2. Moving Least-Squares Method. Let:

- (1)  $\mathcal{D}$  be a bounded domain in  $\mathbb{R}^d$ .
- (2)  $\mathbf{x}_i \in \mathcal{D}$ ,  $i = 0, \dots, m$ ;  $\mathbf{x}_i \neq \mathbf{x}_j$ , if  $i \neq j$ .
- (3)  $f : \mathcal{D} \rightarrow \mathbb{R}$  be a continuous function.
- (4)  $p_i : \mathcal{D} \rightarrow \mathbb{R}$  be continuous functions,  $i = 1, \dots, l$ . The functions  $\{p_1, \dots, p_l\}$  are linearly independent in  $\mathcal{D}$  and let  $\mathcal{P}_l$  be their linear span.
- (5)  $W : (0, \infty) \rightarrow (0, \infty)$  is a strictly positive functions.

Usually the basis in  $\mathcal{P}_l$  is constructed by monomials. For example:  $p_l(\mathbf{x}) = x_1^{k_1} \dots x_d^{k_d}$ , where  $\mathbf{x} = (x_1, \dots, x_d)$ ,  $k_1, \dots, k_d \in \mathbb{N}$ ,  $k_1 + \dots + k_d \leq l - 1$ . In the case  $d = 1$ , the standard basis is  $\{1, x, \dots, x^{l-1}\}$ .

Following [5], [6], [7], [8], we use the following definition. The *moving least-squares approximation* of order  $l$  at a fixed point  $\mathbf{x}$  is the value of  $p^*(\mathbf{x})$ , where  $p^* \in \mathcal{P}_l$  is minimizing the least-squares error

$$\sum_{i=1}^m W(\|\mathbf{x} - \mathbf{x}_i\|) (p(\mathbf{x}) - f(\mathbf{x}_i))^2 \quad (4)$$

among all  $p \in \mathcal{P}_l$ .

The approximation is “local” if weight function  $W(s)$  is fast decreasing as  $s$  tends to infinity. Interpolation is achieved if  $W(0) = \infty$ . We

define additional function  $w : [0, \infty) \rightarrow [0, \infty)$ , such taht:

$$w(s) = \begin{cases} \frac{1}{W(s)}, & \text{if } (s > 0) \text{ or } (s = 0 \text{ and } W(0) < \infty), \\ 0, & \text{if } (s = 0 \text{ and } W(0) = \infty). \end{cases}$$

Some examples of  $W(s)$  and  $w(s)$ ,  $s \geq 0$ :

$$\begin{aligned} W(s) &= e^{-\alpha^2 s^2} && \text{exp-weight,} \\ W(s) &= s^{-\alpha^2} && \text{Shepard weights,} \\ w(s) &= s^2 e^{-\alpha^2 s^2} && \text{McLain weight,} \\ w(s) &= e^{\alpha^2 s^2} - 1 && \text{see Levin's works.} \end{aligned}$$

Here and below: the superscript  $^t$  denotes transpose of matrix;  $I$  is the identity matrix.

Let us introduce the matrices:

$$E = \begin{pmatrix} p_1(\mathbf{x}_1) & p_2(\mathbf{x}_1) & \cdots & p_l(\mathbf{x}_1) \\ p_1(\mathbf{x}_2) & p_2(\mathbf{x}_2) & \cdots & p_l(\mathbf{x}_2) \\ \vdots & \vdots & & \vdots \\ p_1(\mathbf{x}_m) & p_2(\mathbf{x}_m) & \cdots & p_l(\mathbf{x}_m) \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} p_1(\mathbf{x}) \\ p_2(\mathbf{x}) \\ \vdots \\ p_l(\mathbf{x}) \end{pmatrix}$$

$$D = 2 \begin{pmatrix} w(\|\mathbf{x} - \mathbf{x}_1\|) & 0 & \cdots & 0 \\ 0 & w(\|\mathbf{x} - \mathbf{x}_2\|) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & w(\|\mathbf{x} - \mathbf{x}_m\|) \end{pmatrix}.$$

Through the article, we assume the following conditions (H1):

- (H1.1)  $1 \in \mathcal{P}_l$ .
- (H1.2)  $1 \leq l \leq m$ .
- (H1.3)  $\text{rank}(E^t) = l$ .
- (H1.4)  $w$  is a smooth function.

**Theorem 2.1** (see [6]). *Let the conditions (H1) hold true.*

*Then:*

- (1) *The matrix  $E^t D^{-1} E$  is non-singular.*
- (2) *The approximation defined by the moving least-squares method is*

$$\hat{L}(f) = \sum_{i=1}^m a_i f(\mathbf{x}_i), \quad (5)$$

where

$$\mathbf{a} = A_0 \mathbf{c} \quad \text{and} \quad A_0 = D^{-1} E (E^t D^{-1} E)^{-1}. \quad (6)$$

(3) If  $w(0) = 0$ , then the approximation is interpolatory.

### 3. B-SPLINE CURVE AS A MINIMUM OF MOVING LEAST-SQUARES ERROR

Using the definitions and notations introduced in Section 2, our goal is to prove the following theorem.

**Theorem 3.1.** *Let:*

- (1)  $d = 1$ ,  $n, r \in \mathbb{Z}_+$ ,  $r \leq n + 1$ ,  $f : [0, n + r] \rightarrow \mathbb{R}$  be a continuous function.
- (2)  $\mathbf{p}_i = (i, f(i))$ ,  $i = 0, \dots, n + r$ .
- (3) Let  $\boldsymbol{\gamma}(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix}$  be the b-spline of order  $r$  and knot vector  $\{t_i = i : i = 0, \dots, n + r\}$ .

Then  $\gamma_1(t) = t - 2$ ,  $t \in [0, n + r]$  and there exists a weight function  $W$ , such that

$$\gamma_2(x) = \hat{L}(f)(x), \quad x \in [r - 1, n + 1],$$

where  $\hat{L}(f)(x)$  is the approximation defined by the moving least-squares method for the data  $\{\mathbf{p}_i : i = 0, \dots, n + r\}$ .

*Proof.* We will prove the theorem for the cubic splines, i.e.  $r = 4$ . The proof for the different orders is similar.

From conditions (2) and (3), if we set  $x_i = i$ , then  $t_i = x_i$ ,  $i = 0, \dots, n$  and hence  $\gamma_1(t) = t$ .

The b-spline curve of order 4, defined using knots  $\{t_i = i : i = 0, \dots, n + r\}$ , is

$$\gamma_2(t) = \sum_{i=0}^n B_{i,4}(t)f(x_i), \quad x \in [t_{r-1}, t_{n+1}] \equiv [3, n + 1].$$

The following properties of b-spline basis functions  $B_{i,j}(t)$  are well known:

$\vdots$	$\vdots$	
$t_{i_0-2} < t_{i_0-1}$ :	$B_{i_0-2,1}(t) = 0$	$\vdots$
	$B_{i_0-2,2}(t) = 0$	$\vdots$
$t_{i_0-1} < t_{i_0}$ :	$B_{i_0-1,1}(t) = 0$	$B_{i_0-2,3}(t) > 0$
	$B_{i_0-1,2}(t) > 0$	$B_{i_0-2,4}(t) > 0$
$t_{i_0} < t < t_{i_0+1}$ :	$B_{i_0,1}(t) = 1$	$B_{i_0-1,3}(t) > 0$
	$B_{i_0,2}(t) > 0$	$B_{i_0-1,4}(t) > 0$
$t_{i_0+1} < t_{i_0+2}$ :	$B_{i_0+1,1}(t) = 0$	$B_{i_0,3}(t) > 0$
	$B_{i_0+1,2}(t) = 0$	$\vdots$
$t_{i_0+2} < t_{i_0+3}$ :	$B_{i_0+2,1}(t) = 0$	$\vdots$
$\vdots$	$\vdots$	

TABLE 1. Cox-de Boor recursion algorithm

(BS-0) By the direct calculation (see formulas (1), (2), and the schema illustrated in Table 1,  $r = 4$ ):

$$B_{i,4}(t) = \begin{cases} 0, & \text{if } t < i, \\ \frac{1}{6}(t-i)^3, & \text{if } i \leq t < i+1, \\ -\frac{2}{3}(t-i-1)^3 + \frac{1}{6}(t-i)^3, & \text{if } i+1 \leq t < i+2, \\ (t-i-2)^3 - \frac{2}{3}(t-i-1)^3 \\ + \frac{1}{6}(t-i)^3, & \text{if } i+2 \leq t < i+3, \\ -\frac{2}{3}(t-i-3)^3 + (t-i-2)^3 \\ - \frac{2}{3}(t-i-1)^3 + \frac{1}{6}(t-i)^3, & \text{if } i+3 \leq t < i+4, \\ 0, & \text{if } i+4 \leq t. \end{cases}$$

(BS-1) If  $t \in (i, i+j)$ , then  $B_{i,j}(t) > 0$ .

(BS-2) If  $t \in [0, i] \cup [i+j, n+j]$ , then  $B_{i,j}(t) = 0$ .

(BS-3)  $\sum_{i=0}^n B_{i,j}(t) = 1$ , for any  $t \in (j-1, n+1)$ .

(BS-4)  $B_{i,j}(t)$  has  $C^{j-2}$  continuity at each knot.

(BS-5) By the simple substitutions in the formulas in (BS-0):

$$B_{i,j}(t+i) = B_{k,j}(t+k), \quad t \in (0, j),$$

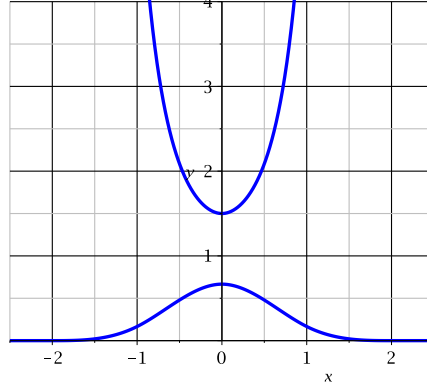


FIGURE 1. The graphics of  $W(x)$ ,  $x \in (-2.5, 2.5)$  and  $w(x)$ ,  $x \in (-1, 1)$

$$B_{i-2,j}(t) = B_{i,j}(t+2), \quad t \in (i-2, i+j-2).$$

Let  $t$  be a fixed point in the interval  $(3, n+1)$  and  $i_0$  be an integer such that  $3 \leq i_0 < t < i_0 + 1 \leq n+1$ . Then

$$\gamma_2(t) = \sum_{i=0}^n B_{i,4}(t)f(i) = \sum_{i=i_0-3}^{i_0} B_{i,4}(t)f(i). \quad (7)$$

because if  $i = 1, \dots, i_0 - 4, i_0 + 1, \dots, n$ , then  $B_{i,4}(x) = 0$ , see (BS-1) and (BS-2).

On the other hand, let us consider the moving least-squares problem for the given data  $\{(i, f(i)) : i = 0, \dots, n+4\}$ . Let us set  $l = 1$ , and

$$W(x) = \begin{cases} 0, & \text{if } x < -2, \\ \frac{1}{6}(x+2)^3, & \text{if } -2 \leq x < -1, \\ -\frac{1}{2}x^3 - x^2 + \frac{2}{3}, & \text{if } -1 \leq x < 0, \\ \frac{1}{2}x^3 - x^2 + \frac{2}{3}, & \text{if } 0 \leq x < 1, \\ -\frac{1}{6}(x-2)^3, & \text{if } 1 \leq x < 2, \\ 0, & \text{if } 2 \leq x, \end{cases}$$

see Figure 1.

Then for any  $i = 0, \dots, n$ , we have (see also (BS-5)):

$$W(|x|) = W(x) = B_{i,4}(x+i+2) = B_{i-2,4}(x+i), \quad x \in [-2, 2].$$

Hence  $W(|x-i|) = B_{i-2,4}(x)$ ,  $x \in [i-2, i+2]$ .

The least-squares error is (see also conditions (H1):  $p_1(x) = 1$ , so  $p(x) = p$  has to be a constant)

$$\sum_{i=1}^{n+r} W(x-i) (p-f(i))^2 = \sum_{i=i_0-1}^{i_0+2} W(x-i) (p-f(i))^2,$$

because  $W(x-i) > 0$ , iff  $i_0 - 1 \leq i \leq i_0 + 2$ .

It is not hard to compute

$$E = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{c} = (1),$$

$$D = 2 \begin{pmatrix} w(|x - (i_0 - 1)|) & 0 & 0 & 0 \\ 0 & w(|x - i_0|) & 0 & 0 \\ 0 & 0 & w(|x - (i_0 + 1)|) & 0 \\ 0 & 0 & 0 & w(|x - (i_0 + 2)|) \end{pmatrix},$$

where  $w(x) = \frac{1}{W(x)}$ , and

$$\begin{aligned} E^t D^{-1} E &= \frac{1}{2} \left( \frac{1}{w(|x - (i_0 - 1)|)} + \frac{1}{w(|x - i_0|)} + \frac{1}{w(|x - (i_0 + 1)|)} \right. \\ &\quad \left. + \frac{1}{w(|x - (i_0 + 2)|)} \right) \\ &= \frac{1}{2} (B_{i_0-3,4}(x) + B_{i_0-2,4}(x) + B_{i_0-1,4}(x) + B_{i_0,4}(x)) \\ &= \frac{1}{2}, \quad \text{because } x \in [i_0, i_0 + 1], \\ \mathbf{a} &= D^{-1} E (E^t D^{-1} E)^{-1} \mathbf{c} = \begin{pmatrix} B_{i_0-3,4}(x) \\ B_{i_0-2,4}(x) \\ B_{i_0-1,4}(x) \\ B_{i_0,4}(x) \end{pmatrix}. \end{aligned}$$

Hence, by Theorem 2.1, we have

$$\hat{L}(f)(x) = \sum_{i=1}^4 a_i f(x_i) = \sum_{i=i_0-3}^{i_0} B_{i,4}(x) f(i),$$

i.e. we received b-spline (7). □

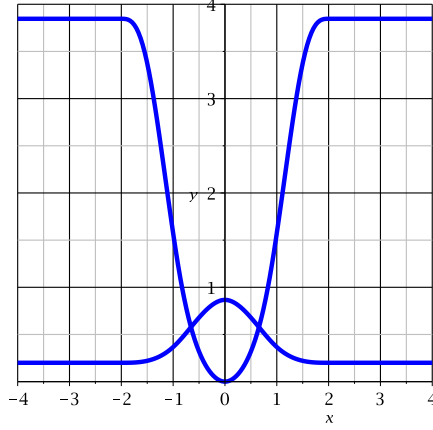


FIGURE 2. The graphics of  $\widetilde{W}(x)$  and  $\widetilde{w}(x)$ ,  $x \in (-4, 4)$

*Remark 1.* Using the method, illustrated in the proof of Theorem 3.1, it is not difficult to generalize the result in whole interval:

$$\gamma_2(x) = \sum_{i=0}^n W(x-i)f(i), \quad x \in [r-1, n].$$

See Example 4.1 in Section 4.

*Remark 2.* Using mentioned in Subsection 1.2, Levin's approach (i.e. working with weight-function  $w(x)$ , such that  $w(x_i) = 0$ ) in moving least-squares method, it is not difficult to receive interpolation. Let

$$\widetilde{W}(x) = W(x) + \delta, \quad x \in \mathbb{R},$$

where  $\delta > 0$ . Then  $\widetilde{W}(x) \geq \delta > 0$ , for any  $x \in \mathbb{R}$  and  $\max\{\widetilde{W}(x) : x \in \mathbb{R}\} = \widetilde{W}(0) = \frac{2}{3} + \delta$ . Let moreover

$$\widetilde{w}(x) = \frac{1}{W(x) + \delta} - \frac{3}{2 + 3\delta}, \quad x \in \mathbb{R}.$$

Then:  $\widetilde{w}(x) > 0$ , for any  $x \in \mathbb{R} \setminus \{0\}$  and  $\min\{\widetilde{w}(x) : x \in \mathbb{R}\} = \widetilde{w}(0) = 0$ , see Figure 2.

In this case the method used in the proof of Theorem 2.1 produces interpolation.

See Example 4.2 in Section 4.

#### 4. SOME EXAMPLES. CASE OF INTERPOLATION

**Example 4.1.** Consider the following example of control points:

$$\Xi_0 = \{(x_i, f(x_i)) : x_i = i,$$



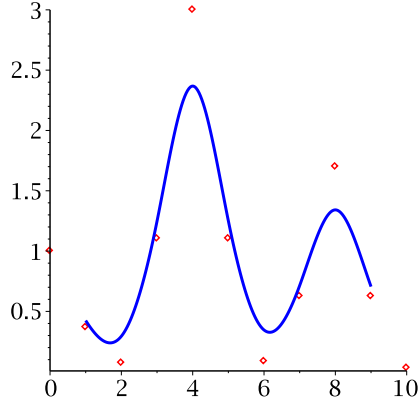


FIGURE 3. The plots of data-set  $\Xi_0$  and cubic b-spline in  $[1, 9]$

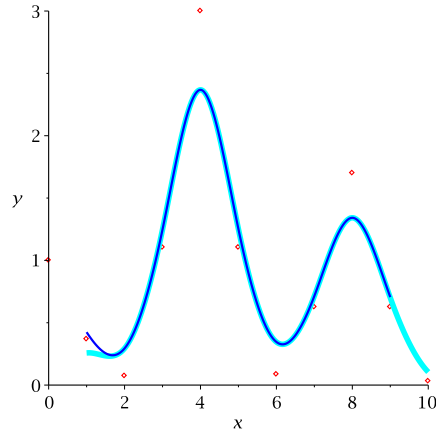


FIGURE 4. The plots of data-set  $\Xi_0$  (red dots), cubic b-spline in  $[1, 9]$  (blue curve) and moving least square approximation in  $[2, 10]$  (cyan bold curve)

$$f(x) = e^{-x^2} + 3e^{-(x-4)^2} + 1.7e^{-(x-8)^2}, \quad i = 0, \dots, 10\}.$$

Here  $n = 10$ ,  $r = 4$ , knots:  $t_i = i$ ,  $i = 1, \dots, 14$ . The control points and cubic b-spline  $\gamma(x)$  are illustrated on Figure 3. Here we used standart Maple expression **BSplineCurve**, see [9].

Using Remark 1 it is easy to construct the b-spline curve in whole interval  $[2, 10]$  – see Figure 4.

Let us apply Theorem 3.1 in the interval  $(5, 6)$ , for example. We have  $W(x) = B_{5-2,4}(x + 5 + 2)$  and moreover

$$W(x) > 0, \quad i = 4, 5, 6, 7; \quad W(x) = 0, \quad i = 0, 1, 2, 3, 8, 9, 10.$$

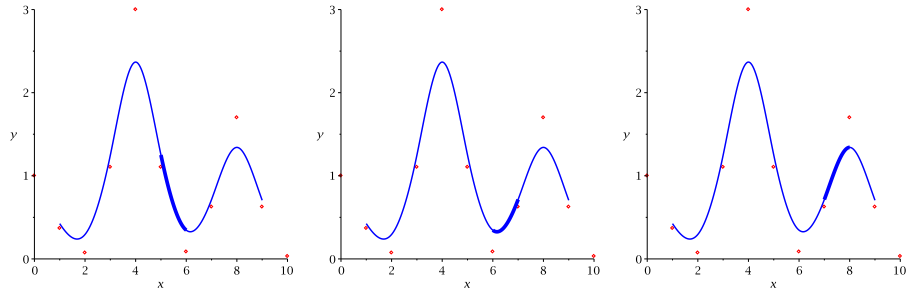


FIGURE 5. The plots of b-spline in  $[5, 6]$ ,  $[6, 7]$ , and  $[7, 8]$ , respectively (blue bold curve)

Following the proof of theorem: let  $w(s) = \frac{1}{W(s)}$ . Let  $x \in (5, 6)$  be a fixed point, then:

$$\begin{aligned} E^t D^{-1} E &= \frac{1}{2} \left( \frac{1}{w(|x-4|)} + \frac{1}{w(|x-5|)} + \frac{1}{w(|x-6|)} + \frac{1}{w(|x-7|)} \right) \\ &= \frac{1}{2}, \end{aligned}$$

see (BS-3). Moreover

$$\mathbf{a} = D^{-1} E (E^t D^{-1} E)^{-1} \mathbf{c} = D^{-1} E \mathbf{2} = \begin{pmatrix} B_{2,4}(x) \\ B_{3,4}(x) \\ B_{4,4}(x) \\ B_{5,4}(x) \end{pmatrix}.$$

So, we receive the classical b-spline formula in the interval  $(5, 6)$ :

$$\gamma_2(x) = \sum_{i=2}^5 W(x-i) f(i) = \sum_{i=2}^5 B_{i,4}(x) f(i).$$

The b-splines in the intervals  $[5, 6]$ ,  $[6, 7]$ , and  $[7, 8]$  are plotted on Figure 5, respectively.

**Example 4.2.** To construct the interpolation in the interval  $[x_0, x_{10}]$ , following Remark 2, let us set  $\delta = 0.1$ , for example. Then

$$\tilde{w}(x) = \frac{1}{W(x) + 0.1} - \frac{3}{5}, \quad x \in \mathbb{R}.$$

Applying the moving least-squares method (i.e. applying Remark 2 and Theorem 2.1 at each point  $x = l/100$ ,  $l = 1, \dots, 1000$ ) we received the function presented in Figure 6.

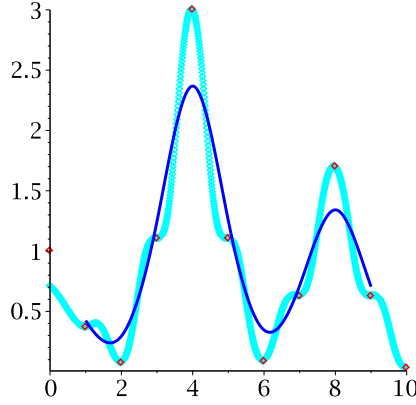


FIGURE 6. The plots of data (red dots), b-spline (blue), and interpolation (cyan, bold)

## 5. B-SPLINE SURFACES

Let  $\{\mathbf{p}_{ij} \in \mathbb{R}^3 : i, j = 0, \dots, n\}$  be a set of  $(n+1)^2$  control points.

Let  $r$  be an integer,  $1 \leq r \leq n+1$  (the order of spline).

We will use again the uniform knots, without multiplicity:  $u_i = v_i = i$ ,  $i = 0, \dots, n+r$ . Then, the corresponding b-spline surface of order  $r$  is given by

$$\mathbf{r}(u, v) = \sum_{i=0}^n \sum_{j=0}^n B_{i,r}(u) B_{j,r}(v) \mathbf{p}_{ij}.$$

Arguments similar to the proof of Theorem 3.1, yield to the following result.

**Theorem 5.1.** *Let:*

- (1)  $d = 2$ ,  $n, r \in \mathbb{Z}_+$ ,  $r \leq n+1$ ,  $f : [0, n+r] \times [0, n+r] \rightarrow \mathbb{R}$  be a continuous function.
- (2)  $\mathbf{p}_{ij} = (i, j, f(i, j))$ ,  $i = 0, \dots, n+r$ .
- (3) Let  $\gamma(u, v) = \begin{pmatrix} u \\ v \\ \gamma_3(u, v) \end{pmatrix}$  be the b-spline of order  $r$  and knots  $\{(u_i, v_j) = (i, j) : i, j = 0, \dots, n+r\}$ .

Then there exists a weight function  $W(x, y)$ , such that

$$\gamma_3(x, y) = \hat{L}(f)(x, y), \quad (x, y) \in [r-1, n] \times [r-1, n].$$

**Example 5.1.** As an example, consider the following data

$$\begin{aligned} \Xi_0 &= \{(x_i, y_j, f(x_i)) : x_i = i, y_j = j, \\ &\quad f(x_i, y_j) = e^{-x^2} + 3e^{-(y-1)^2} + e^{-(x-6)^2 - (y-6)^2}, i = 0, \dots, 10\}. \end{aligned}$$

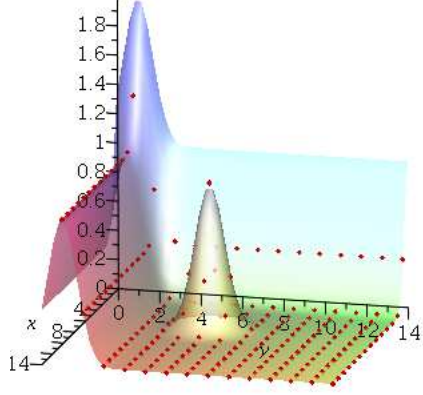


FIGURE 7. Set of knots and plot of function  $f(x, y)$ ,  $x, y \in [0, 14]$

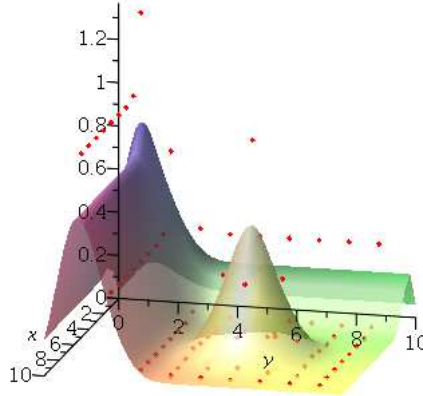


FIGURE 8. B-spline surface  $\gamma_3(u, v)$  in  $[1, 9] \times [1, 9]$

The set of control points  $\mathbf{p}_{ij} = \begin{pmatrix} x_i \\ y_j \\ f(x_i, y_j) \gamma_2(t) \end{pmatrix}$  and function  $f(x, y)$  are plotted in Figure 7.

Using the formula

$$\hat{L}(f)(x, y) = \sum_{i=0}^n \sum_{j=0}^n \mathbf{p}_{ij} W(x - i) W(x - j).$$

we receive the plot of b-spline surface  $\gamma_3(u, v)$  in  $[1, 9] \times [1, 9]$ , see Figure 8.

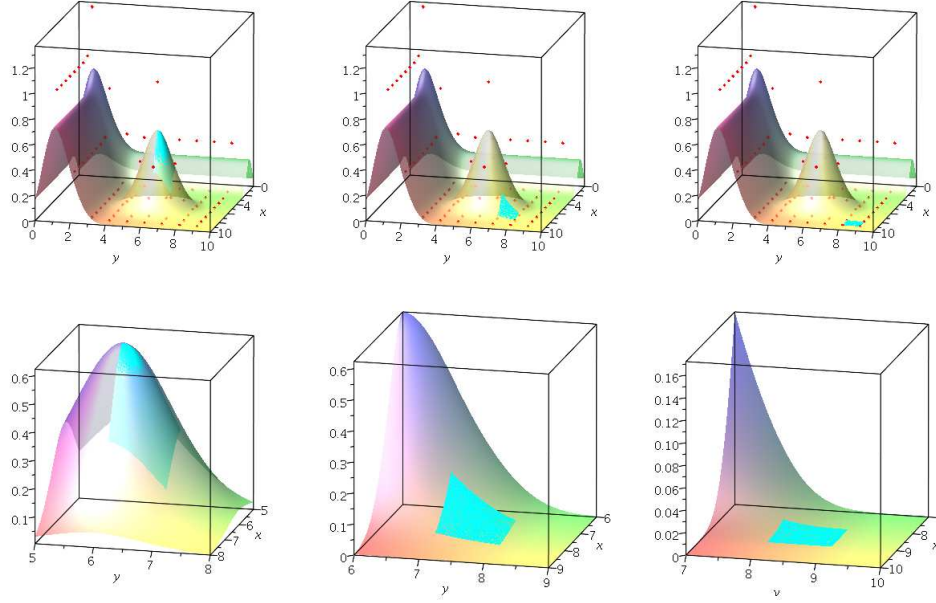


FIGURE 9. B-splines in  $[i, i+1] \times [i, i+1]$ ,  $i = 6, 7, 8$  and corresponding zoomed segments

If we need the plot of b-spline only in the segment  $[i_0, i_0+1] \times [j_0, j_0+1]$ , then

$$\hat{L}(f)(x, y) = \sum_{i=i_0-3}^{i_0} \sum_{j=j_0-3}^{j_0} \mathbf{p}_{ij} W(x-i)W(y-j),$$

see Figure 9.

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